Advanced algorithms

Exercise sheet #8 (Solutions) — Primal-Dual & approximation algorithms

November 23, 2022

Exercice 1 (Multicut Problem in Trees)**.**

Let $G = (V, E)$ be an undirected weighted graph with integer weights $c_e \geq 0$ for each edge $e \in E$, and let $\{(s_1, t_1), \dots, (s_k, t_k)\}$ be k specific pairs of vertices.
A *multicut* is a set of edges F such that for all i, s_i and t_i are in different connected components of $G' = (V, E \setminus F)$. The weight $c(F)$ of F is the sum of the weight of its elements. The goal of *minimum multicut* problem is to find a multicut of minimum weight.
Recall in exercise sheet #6 we gave a 2-approximation algorithm for a slightly different problem called *multiterminal cut* where we were given a set of k vertices instead of pairs of vertices. Here, we need to deal with the fact that there may be paths connecting s_i and s_j or s_i and t_j for $i \neq j$.
The multicut problem is known to be NP-hard (you need not to prove it), so there is no hope in this session to find a polynomial time algorithm to solve it. The goal of this exercise is to use a Primal-Dual method to design an approximation algorithm in the special case where G is assumed to be a tree.

(a) Let $\mathcal{P}_i$ be the set of all paths P joining s_i and t_i for some $1 \leq i \leq k$. What is the cardinality of $\mathcal{P}_i$?

Solution — G is a tree, so is connected and acyclic. Therefore, for any pair of distinct vertices v and w, there is a unique *path joining them. In particular, for all i, $|\mathcal{P}_i| = 1$.*

Consider the following linear progam:

$$\begin{aligned} &\text{minimize} \sum_{e \in E} c_e x_e \\ &\text{subject to} \sum_{e \in P} x_e \geqslant 1, \quad \forall P \in \mathcal{P}_i, 1 \leq i \leq k, \\ &x_e \geq 0, \quad \forall e \in E, \end{aligned}$$

with the optimal value λ^* and an optimal solution (x_e^*).

(b) Let F^* be a minimum multicut of G. Show that $c(F^*) \geq \lambda^*$.

Solution — If $x_e^* \in \{0, 1\}$, then it is easy to build a minimum multicut. Let

$$F \doteq \{e \in E \mid x_e = 1\}.$$

The constraint $\sum_{e\in P} x_e \geq 1$ ensures that for each pair (s_i, t_i) we take one edge in the unique path joining them, thus disconnecting s_i from t_i, i.e. *that F is indeed a multicut and λ^* is exactly the weight of F.*

Therefore, multicut correspond to integer solutions of this linear program. When we allow x_e to take arbitrary positive real values, the minimum weight of a multicut cannot be larger. Hence $c(F^*) \geq \lambda^*$.

(c) We introduce a variable f_i for each pair (s_i, t_i). Give the dual of this linear program.

Solution —

$$\begin{aligned} \textit{maximize} & \sum_{i=1}^{k} f_i \\ \textit{subject to} & \sum_{i:\, e\in P_i} f_i \leqslant c_e, \quad \forall e \in E \\ & f_i \geq 0, \quad \forall i, \end{aligned}$$

where P_i is the unique path joining s_i and t_i.

Suppose we root the tree G at an arbitrary vertex r. Let *depth*(v) be the number of edges on the path from v to r for any vertex v. The depth of the root is 0. belong to the same path from a vertex to the root, if e_1 occurs before e_2 on this path, e_1 is said to be deeper than e_2. Denote by $lca(s_i, t_i)$ the *least common ancestor* of s_i and t_i, *i.e.* the vertex v on the path from s_i to t_i whose depth is minimum.

Consider the following algorithm:

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1 Set $f_i \leftarrow 0$ for all $i = 1, \dots, k$ and $F \leftarrow \emptyset$
2 For each $v \in V$, in non-increasing order of depth, do:
    (i) For each $i$ such that $v = lca(s_i, t_i)$, increase $f_i$ such that some inequality
        constraint in the dual becomes an equality (saturation of edges).
    (ii) Add to $F$ all edges that were saturated.
3 Let $e_1, \dots, e_l$ be the list of edges in $F$ in the order they were added.
4 Reverse delete For $j = l$ downto 1 do:
    (i) If $F \setminus \{e_j\}$ is a multicut in $G$, then remove $e_j$ from $F$.
5 Output $F$.
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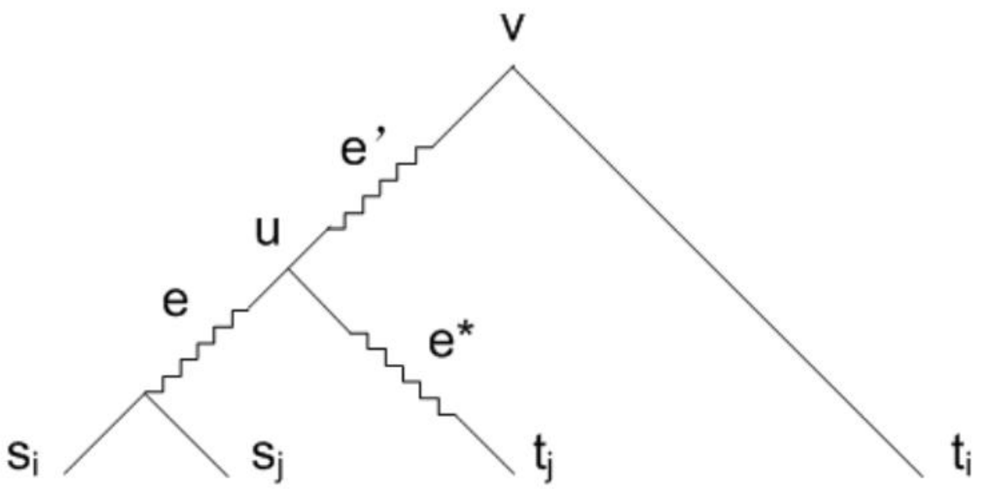
(d) Show that the set F output by the algorithm is a multicut of G, and (f_i) is a feasible dual solution.

Solution — We start from 0 which is a feasible solution of the dual, and maintain all the constraints during the algorithm. Therefore at the end, (f_i) is still feasible. We just need to prove that F is a multicut before deletion. If we remove F from E, there cannot be any (s_i, t_i) path, otherwise it means that when we considered $lca(s_i, t_i)$ there exists an edge that hasn't been saturated, therefore f_i could have been increased, this contradicts the greedy paradigm.

(e) Show that for every pair (s_i, t_i) such that $f_i > 0$, F contains at most two edges from the path $s_i \to t_i$.

Solution — Let (s_i, t_i) be such a pair and let $v = lca(s_i, t_i)$. Consider the two subpaths $T_1 = s_i \to v$ and $T_2 = v \to t_i$. We will prove that at most one edge is picked on each of them. By the way of contradiction, suppose that two edges e and e' are picked from T_1 (the same argument also works for T_2). Without loss of generality, assume e is closer to s_i than e', i.e. *e appears before e' on the path $s_i \to v$. Consider the time during the reverse removal when e is tested. Since e is not discarded, there must be another pair (s_j, t_j) such that e is the only edge picked on the path $s_j \to t_j$. Let $u = lca(s_j, t_j)$. Since $e' \notin s_j \to t_j$, u must be deeper than v and so is processed before v in the algorithm. After u is processed, F contains an edge e^* from the path $s_j \to t_j$.*

Since $f_i > 0$, e must be added during, or after, the iteration in which v is processed, therefore e must be added in F after *e^*. So e^** must *be in F when e is being tested for deletion. In particular, $e \neq e^*$. This contradicts the fact that at this moment e is the only edge of F on the path $s_j \to t_j$.*



(f) Deduce that this algorithm achieves an approximation factor of 2 for the multicut problem in trees.

Solution — By the previous questions, the set F output by the algorithm is a multicut. Since each edge of F is saturated (we only pick saturated edges), then

$$\forall e \in F, c_e = \sum_{i:\, e \in P_i} f_i.$$

The weight of F is by definition

$$c(F) = \sum_{e \in F} c_e.$$

Therefore the following holds:

$$\begin{aligned}
c(F) &= \sum_{e\in F}\sum_{i=1}^{k} f_i \mathbb{1}_{e\in P_i} \\
&= \sum_{i=1}^{k} f_i \sum_{e\in F} \mathbb{1}_{e\in P_i} \\
&= \sum_{i=1}^{k} f_i |F\cap P_i| \\
&\le 2\sum_{i=1}^{k} f_i.
\end{aligned}$$

Let (f_i^*) be an optimal solution of the dual. Since (f_i) is a feasible solution,

$$\sum_{i=1}^{k} f_i \le \sum_{i=1}^{k} f_i^* = \lambda^* \le c(F^*).$$

All in all,

$$c(F^*) \le c(F) \le 2\cdot c(F^*).$$

Exercice 2 (Maximizing Ad-Auctions Revenue (Pâle 2015))**.**

The aim of this exercice is to analyze a system of one shot auctions used for selling advertisment slots on the web. The auction we want to study is organized by a provider that needs to decide every day to which of his n clients he will sell each of his m advertisment slots. To make his choice, the provider knows the daily budget $B(i)$ of each client $i = 1,\dots,n$ and the bids $b(i,j)$, $i = 1,\dots,n$, $j = 1,\dots,m$ made independantly by each client for each slot.

In the *offline* variant, the $B(i)$ and the $b(i,j)$ are all known from the beginning of the day and the problem is for the provider to determine the best possible assignment of the slots to the clients. Beware that it is not a standard "best bid wins" auction: with 2 clients with budget $B(1) = B(2) = 3$, if the first client offers $b(1,1) = 3$ and $b(1,2) = 2$ and the second offers $b(1,1) = 2$ and $b(1,2) = 0$, it is more interesting to sell slot 1 to client 2 than to client 1 in order to be able to sell slot 2 to client 1.

(a) Show that with 3 clients having each a budget $B(i) = 4$ and 6 slots, if the bids of clients are given by the following matrix $b(i,j)$:

$$\begin{pmatrix} 3 & 3 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$$

then in the optimal solution none of the slot is sold to the client proposing the highest bid.

Solution — Consider the situation where 2 slots are sold at cost 2 to each of the bidders. This is a valid solution, and it uses all the available budget. Therefore, it is necessary optimal. On the other hand, if a slot is sold to the client with the highest bid, this client cannot spend their remaining budget.

(b) Give a modelization of the offline problem as an integer linear program. Use variables $y(i,j)$ in $\{0,1\}$ to indicate whether slot j is sold to client i.

Solution —

$$\begin{aligned}
&\textit{maximize} \sum_{i,j} b(i,j)y(i,j)\\
&\textit{subject to}\\
&\qquad \forall 1 \le i \le n\colon \sum_j b(i,j)y(i,j) \le B(i)\\
&\qquad \forall 1 \le j \le m\colon \sum_i y(i,j) \le 1\\
&\qquad \forall i,j\colon y(i,j) \in \{0,1\},
\end{aligned}$$

where the constraints on i prevents the i-th client from overspending, and the constraints on j prevents the j-th slot to be sold to more than one client.

(c) Give an interpretation of the real linear program $\mathcal{P}$ in which the constraint $y(i,j) \in \{0,1\}$ on variables is replaced by $y(i,j) \in \mathbb{R}$, $y(i,j) \geqslant 0$?

Solution — In the primal problem, we are asked to maximize the total revenue. By relaxing the integer constraint, we authorize fractional part of slots to be sold to the bidders.

(d) Write the dual linear program $\mathcal{D}$ of the linear program $\mathcal{P}$: use variables $x(i)$ indexed by clients and variables $z(j)$ indexed by slots. You are NOT asked to give an interpretation to $\mathcal{D}$.

Solution — The constraints of the primal problem are inequalities, so we work on the dual with non negative variables. Similarly, the variables of $\mathcal{P}$ are non negative, so the constraints on the dual are inequalities. The dual problem is then

$$\begin{aligned}
&\textit{minimize} \sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m} z(j)\\
&\textit{subject to}\\
&\qquad \forall i,j\colon \quad b(i,j)x(i) + z(j) \ge b(i,j)\\
&\qquad \forall i,j\colon \quad x(i) \ge 0, z(j) \ge 0.
\end{aligned}$$

In the *online* variant, the $B(i)$ are known from the beginning but slots become available one after the other in the order $j = 1, \dots, m$. When slot j becomes available, the clients make their bids $b(i,j)$, $i = 1, \dots, n$ and the provider has to decide immediately to whom he sells the slot.

(e) Give a pair of instances of the problem with $n = m = 2$ that shows that no online algorithm can reach the optimum solution in all cases.

Solution — We come back to the example of the $b(i,j)$ in the introduction with budget $B(i) = 4$ and compare to the case where $b'(i,1) = b(i,1)$ but $b'(1,2) = 1$ and $b'(2,2) = 1$:

If we want to reach the optimum, a different decision needs to be taken for the sale of the first slot in both cases, whereas the same piece of information is available at that time.

Consider the following algorithm, using variables $x(i)$, $y(i,j)$ and $z(j)$ for $i = 1,\dots,n$ and $j = 1,\dots,m$:

1. Initially, for each bidder i, set $x(i) \leftarrow 0$.
2. For all $j = 1,\dots,m$, when slot j becomes available, the provider tries to sell it to the client i_j that maximises $b(i,j)(1-x(i))$.
   (i) if $x(i_j) < 1$, then
   - Sell the slot to client i_j at a price equal to the minimum between his bid $b(i_j,j)$ and his remaining budget, set $y(i_j,j) \leftarrow 1$ and for all $i \neq i_j$, $y(i,j) \leftarrow 0$.
   - Set $z(j) \leftarrow b(i_j,j)(1-x(i_j))$.
   - Update
     $$x(i_j) \leftarrow x(i_j)\left(1 + \frac{b(i_j,j)}{B(i_j)}\right) + \frac{b(i_j,j)}{(c-1)\cdot B(i_j)}$$
     where $c > 1$ is a constant to be determined later.

   (ii) Otherwise the provider does not sell this slot and set: $z(j) \leftarrow 0$, $y(i,j) \leftarrow 0$ for all i.

The aim of the rest of the exercice is to prove that it is possible to choose the constant c so that this algorithm be α-competitive for some α.

(f) Show that after selling the k-th slot, variables $x(i)$, $i = 1,\dots,n$ and $z(j)$, $j = 1,\dots,k$ satisfy the contraints of the dual program $\mathcal{D}$.

Solution — Upon selling the j-th slot,

- *Either $x(i_j) \geq 1$ and for all i the constraint related to i and j is satisfied because*
  $$b(i,j)(1-x(i)) \leq b(i_j,j)(1-x(i_j)) \leq 0 = z(j).$$
- *Or $x(i_j) < 1$ and the choice of $z(j)$ ensures that*
  $$b(i,j)(1-x(i)) \leq b(i_j,j)(1-x(i_j)) = z(j).$$

During the remaining part of the algorithm, $x(i)$ cannot decrease, and therefore the previous constraint is always fulfilled.

(g) Deduce from the previous question that the final values of the variables $x(i)$ and $z(i)$ satisfy

$$O^* \leqslant \sum_i B(i)x(i) + \sum_j z(j)$$

where O^* denotes the optimal profit of the provider in the *offline* variant of the problem with the same $B(i)$ and $b(i,j)$.

Solution — The duality theorem states that if $x(i), z(j)$ are feasible points then the related dual objective is itself greater than the optimum to the integer valued problem.

(h) Show that for all i and j the values $x_j(i)$ and $x_{j+1}(i)$ of the variable $x(i)$ just before and just after the treatment of the jth slot and the final values of the variables $y(i,j)$ and $z(j)$ satisfy

$$\left(\sum_i B(i)(x_{j+1}(i) - x_j(i))\right) + z(j) \leqslant (1 + \tfrac{1}{c-1}) \sum_i b(i,j)y(i,j)$$

and conclude that

$$O^* \leqslant (1 + \tfrac{1}{c-1}) \sum_{i,j} b(i,j)y(i,j).$$

Solution — If the sale is concluded, the sum in the right hand side of the inequality is reduced to $b(i_j, j)$, otherwise to 0. Moreover $x_{j+1}(i) - x_j(i) = 0$ if $i \neq i_j$ and

$$\begin{aligned} B(i_j)(x_{j+1}(i_j) - x_j(i_j)) + z(j) &= \left(b(i_j,j)x_j(i_j) + \frac{b(i_j,j)}{c-1}\right) + b(i_j,j)(1 - x(i_j)) \\ &= b(i_j,j)\left(1 + \frac{1}{c-1}\right). \end{aligned}$$

Summing for all j and using the previous question yields the second inequality.

It could be tempting to conclude here that we have proved the algorithm to be $(1 - \frac{1}{c})$-competitive. However this would not be correct because in some cases the provider charges the client with a lower price than the price $b(i,j)$ he offered. The sum in the right hand side of the previous inequality is the sum of the prices that were offered by the clients for the slots they bought, while we are really interested in the actual total revenue of the provider.

(i) Show that condition $(*)$ below ensures that each client benefits at most once a day from a better price than the one he offered:

If after selling slot k to client i_k the following holds,

$$\sum_{j=1}^{k} b(i_k,j)y(i_k,j) \geqslant B(i_k) \qquad (*)$$

then, from then on, $x(i_k) \geqslant 1$.

Solution — If the provider sells slot k to bidder i_k at a lower price than $b(i_k, k)$ it means by design of the algorithm that

$$B(i_k) - \sum_{j=1}^{k-1} b(i_k,j)y(i_k,j) \leqslant b(i_k,k).$$

condition $(*)$ implies then than $x(i_k) \geqslant 1$ and the algorithm ensures that no slot is sold to i_k.

(j) Show that if we take $c = (1+R_{\max})^{1/R_{\max}}$ with $R_{\max} = \max_{i,j}\left(\frac{b(i,j)}{B(i)}\right)$, then Condition $(*)$ is verified. For this, show by induction that the inequality

$$x(i) \geqslant \frac{1}{c-1}\left(c^{\frac{\sum_j b(i,j)y(i,j)}{B(i)}} - 1\right)$$

remains true during all the execution of the algorithm, using the fact that, for this choice of c,

$$c^{\frac{b(i,j)}{B(i)}} \leqslant 1 + \frac{b(i,j)}{B(i)} \qquad \text{for all } i,j.$$

Solution — Initially, $y(i,j) = 0$ for all i,j and the equality is trivial. Then, $x(i)$ changes only when a slot k is sold to bidder i. By induction, the following holds

$$\begin{aligned}
x_{k+1}(i) &= x_k(i)\left(1+\frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{(c-1)B(i)}\\
&\geqslant \frac{1}{(c-1)}\left(c^{\frac{\sum_{j<k} b(i,j)y(i,j)}{B(i)}} - 1\right)\left(1+\frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{(c-1)B(i)}\\
&= \frac{1}{(c-1)}\left(c^{\frac{\sum_{j<k} b(i,j)y(i,j)}{B(i)}}\left(1+\frac{b(i,k)}{B(i)}\right) - 1\right).
\end{aligned}$$

With the given choice of c, this inequality can be rewritten as

$$\begin{aligned}
x_{k+1}(i) &\geqslant \frac{1}{(c-1)}\left(c^{\frac{\sum_{j<k} b(i,j)y(i,j)}{B(i)}} c^{\frac{b(i,k)}{B(i)}} - 1\right),\\
&= \frac{1}{(c-1)}\left(c^{\frac{\sum_{j\leqslant k} b(i,j)y(i,j)}{B(i)}} - 1\right)
\end{aligned}$$

which is exactly the wanted result, and implies $(*)$.

(k) Deduce from the previous two questions that for all i,

$$\sum_j b(i,j)y(i,j) \leqslant B(i) + \max_j(b(i,j)),$$

and conclude that the provider obtains from this client a gain at least

$$\left(\sum_j b(i,j)y(i,j)\right)\frac{B(i)}{B(i)+\max_j(b(i,j))} \geqslant \left(\sum_j b(i,j)y(i,j)\right)(1-R_{\max})$$

Solution — The first inequality is the immediate consequence of the fact that the bidder has a discount at most once a day. The gain is then

$$\begin{aligned}
\min\left(B(i), \sum_j b(i,j)y(i,j)\right) &= \left(\sum_j b(i,j)y(i,j)\right)\min\left(\frac{B(i)}{\sum_j b(i,j)y(i,j)}, 1\right)\\
&\geqslant \left(\sum_j b(i,j)y(i,j)\right)\frac{B(i)}{B(i)+\max_j(b(i,j))}.
\end{aligned}$$

The last inequality results from the definition of $R_{\max}$.

(l) What is finally the competitivity factor of the algorithm?

Solution — The algorithm is $(1 - \frac{1}{c})(1 - R_{\max})$-competitive, with

$$c = (1 + R_{\max})^{1/R_{\max}}.$$

When $R_{\max} \to 0$, the competitive ratio tends to $1 - 1/e \approx 0.63$.