Advanced algorithms

Exercise sheet #2 – NP-completeness, branch-and-bound, exponential-time algorithms

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Exercice 1 (NP-completeness of HITTING SET)**.** The problem HITTING SET is the following:

Input: A finite collection $\mathcal{C}$ of finite sets of integers; an integer $k \geqslant 0$.
Question: Is there a set $S \subset \mathbb{N}$ such that $\#S \leqslant k$ and $A \cap S \neq \varnothing$ for any $A \in \mathcal{C}$?

Using the NP-complete problem VERTEX COVER, show that HITTING SET is NP-complete.

Exercice 2 (NP-completeness of FEEDBACK ARC SET)**.** Let $G = (V, E)$ be a directed graph. A *feedback arc set* is a subset of E such that the graph $(V, E \setminus C)$ is acyclic.

The problem FEEDBACK ARC SET is the following:

Input: A directed graph G; an integer $k \geqslant 0$.
Question: Does G admit a feedback arc set of cardinality $\leqslant k$?

(a) Show that FEEDBACK ARC SET is in NP.

Given an undirected graph $G = (V, E)$, we define a directed graph $G' = (V', E')$ as follows: $V' = V \times \{0, 1\}$ and $E' = \left\{ \left(x^0, y^1\right) \text{ s.t. } \{x, y\} \in E \right\} \cup \left\{ \left(x^1, x^0\right) \text{ s.t. } x \in V \right\}$, where, for short, we denote x^i the pair $(x, i) \in V'$.

(b) Show that G' has a minimum-cardinality feedback arc set with edges only of type (x^1, x^0).

(c) Show that FEEDBACK ARC SET is NP-complete. *Hint: use* VERTEX COVER.

Exercice 3 (Dynamic programming for TSP)**.** Let $G = (V, E)$ be a complete graph with a weight function $w : E \to \mathbb{R} \cup \{+\infty\}$. The *travelling salesman problem (TSP)* is the problem of computing a minimum-weight cycle in G that goes through all vertices exactly once. The naive solution to solve this problem is the enumeration of all $n!$ cycles (where $n = \#V$).

The goal of this exercise is to do better and to provide an exponential time algorithm.

(a) Let $S \subset V$ containing at least two vertices, and let $s \in S$. For any $t \in S$, we denote by $W(S, t)$ the total weight of a shortest path from s to t, visiting exactly once the elements of S and no other vertex.

Give a recursive expression of $W(S, t)$.

(b) Deduce an algorithm to solve TSP in time $O(n^2 2^n)$ using the principle of *dynamic programming.*

Exercice 4 (WalkSAT)**.** We study a randomized local search procedure to solve 3-SAT problem. It takes as input a 3-SAT instance P and an assigment a of the variables that does not satisfy P:

```
procedure UPDATE(P, a)
    Choose a clause c of P that is not satisfied by a
    Choose randomly one the variables appearing in c
    Flip the assignment of v in a
end procedure
```

For two affectations a and b of the variables of P, let $d(a, b)$ denote the Hamming distance of a and b: this is the number of variables that are not assigned to the same value in a and b.

We consider now a 3-SAT instance P that has a solution s.

(a) Let a be an assignment of the variables of P that does not satisfy P, and let $a' = \textsc{Update}(P, a)$. Show that $d(a', s) = d(a, s) \pm 1$ and that

$$\mathbb{P}\left[d(a', s) = d(a, s) - 1\right] \geqslant \frac{1}{3}.$$

We now consider the following algorithm that applie the procedure Update until finding a solution, or stops with the symbol $\varnothing$ if it has not found anything after N iterations.

```
Intput: a 3-SAT instance P, an assignment a and an integer N
Output: a solution of P or ∅
  procedure WALKSAT(P, a, N)
      for k from 1 to N do
          if a satisfies P then
              return a
          end if
          a ← UPDATE(P, a).
      end for
      return ∅
  end procedure
```

Let $h(a, N)$ be the probability that $\textsc{WalkSAT}(P, a, N)$ returns a solution of P.

(b) Let a be an assignment of the variables and let $\delta = d(a, s)$. Show that

$$h(a, 3\delta) \geqslant \binom{3\delta}{\delta} \left(\frac{1}{3}\right)^{2\delta} \left(\frac{2}{3}\right)^{\delta}.$$

For this, one can compare $h(a, 3\delta)$ with the probability to obtain δ tails and 2δ faces out of 3δ coin flips with a biased coin.

(c) We admit the lower bound $\binom{3\delta}{\delta} \geqslant (3\delta+1)^{-1} \left(\frac{27}{4}\right)^{\delta}$. If a is random and uniformly distributed among all possible assignments of the n variables of P, show that

$$\mathbb{P}\left[\textsc{WalkSAT}(P, a, 3n) \text{ returns a solution of } P\right] \geqslant (3n + 1)^{-1} \left(\frac{3}{4}\right)^{n}.$$

(d) Deduce a probabilistic algorithm that returns with probability $\geqslant \frac{1}{2}$ a solution of satisfiable 3-SAT instance with n variables with $\mathrm{poly}(n) \left(\frac{4}{3}\right)^{n}$ operations.