Advanced algorithms

Exercise sheet #3 – Parametric complexity

October 5, 2022

Exercice 1 (Independent set for planar graphs)**.** Recall the Independent set problem:

Input: A graph $G = (V, E)$; an integer k;
Queston: Is there a subset $S \subseteq V$ such that $\#S \geqslant k$ and for any $a, b \in S$, $\{a, b\}$ is not in E?

It is not known to be FPT with respect to k. However, we will show that it is FPT when restricted to planar graphs.

(a) Show that a planar graph with $n \geqslant 3$ vertices has at most $3n - 6$ edges. *Hint: use Euler's formula for connected planar graphs* $\#vertices + \#faces = 2 + \#edges$ *(which includes the outter face).*

(b) Show that a planar graph has a vertex of degree at most 5.

(c) Deduce a $O(6^k n)$-time algorithm for Independent set restricted to planar graph.

Exercice 2 (Covering points by lines)**.** Consider the NP-complete problem Line cover:

Input: A set P of points in the plane; an integer k;
Question: Is P coverable by k lines?

Show that instances of Line cover admit kernels of size $O(k^2)$. (And therefore, Line cover is FPT with respect to k.)

Exercice 3 (Kernel for Vertex cover)**.** We aim at proving that Vertex cover can be solved with $O(kn + 5^{k/4}k^3)$ operations, where k is the size of the cover and n is the number of vertices of the input graph.

(a) Show that instances of Vertex cover have kernels of size $\leqslant k^2$. *Hint: Consider points with large degree, if any.*

(b) For any graph G, let $\Delta(G)$ be the maximum degree of a vertex of G. Show that Vertex cover restricted to graphs with $\Delta(G) \leqslant 2$ is easily solvable.

(c) Use a *branch and bound* approach to show that Vertex cover is solvable in $O(1.5^k(n+m))$ operations, where n is the number of vertices and m the number of edges.

(d) Conclude.

Exercice 4 (Hamming center problem (Pâle 2019))**.** A *word* of length n on the alphabet A is a sequence of n elements of A. Let $u[i]$ denote the ith letter of u, so that $u = u[1] \dots u[n]$. Given two words u and v of length n, let $d(u, v)$ denote their Hamming distance. It is the number of places at which the two words are different: $d(u, v) = |\{i \in \{1, \dots, n\} \mid u[i] \neq v[i]\}|$.

We consider the problem d-CENTER, inspired from error correcting codes questions:

> Problem d-CENTER
>
> **Input:** k words $w_1,\dots,w_k$, each of length n, and an integer d
> **Question:** is there a word u such that $d(u, w_i) \leqslant d$ for any $i = 1, \dots, k$?

To answer the problem d-CENTER we put the words in a $k \times n$ matrix with entries in A. A column i is *bad* if there are two lines h and ℓ such that $w_h[i] \neq w_\ell[i]$; the column is *good* otherwise.

(a) What can we say about the d-center problem when there are more than kd bad columns?

(b) Assume that we are given a word v and an integer ℓ such that the distance of v to some d-center u of $w_1, \dots, w_k$ is at most ℓ. Assume also that v is not a d-center.

Show how to compute, in linear time, $d + 1$ words $v_1, \dots, v_{d+1}$ such that the distance of one of them to some d-center of $w_1, \dots, w_k$ is at most $\ell - 1$.

(c) Deduce an algorithm for solving d-CENTER with complexity $O(kn + k(d + 1)^{d+1})$.

(d) Is this problem FPT ?