# Advanced algorithms

INF550 – Exercise sheet #1 – Linear programming

September 21, 2022

**Exercice 1** (Shortest path)**.** Given a weighted directed graph $G = (V, E)$ with positive weights, and two distinguished vertices $s$ and $t$, we consider the problem of finding the shortest path from $s$ to $t$ (where the length of a path is the sum of the weights of the edges of the path).

To each vertex $v$ is associated a variable $d(v)$. Consider the following linear problem:

$$\begin{aligned} &\text{maximize } d(t) - d(s) \\ &\text{subject to } \forall (u, v) \in E, d(v) - d(u) \leqslant c(u, v). \end{aligned}$$

(a) Show that for any assignment of the variables $d(v)$ in the feasible set, the difference $d(t)-d(s)$ is a lower bound on the length of the shortest path from $s$ to $t$.

(b) Show that assigning to $d(v)$ the length of the shortest path from $s$ to $v$ is feasible.

(c) Conclude that the optimal value of the linear program is exactly the length of the shortest path from $s$ to $t$.

(d) Write the dual linear program and interpret it.

**Exercice 2** (Max-norm linear regression)**.** A physicist measures a quantity theoretically given by a linear function $y(x) = ax + b$. The results are points $(x_i, y_i)$. He wants to find the best values for the coefficients $a$ and $b$ such that the vertical distance between the points $(x_i, y_i)$ and the line $y = ax + b$ is minimized.

(a) Write this optimization problem as a three-variable linear program.

(b) Write the dual problem.

**Exercice 3** (Preemptive scheduling on parallel computers)**.** A set of computational tasks $\{1, \dots, n\}$ is to be executed simultaneously on $m$ computers. Each task has a duration $p_i$ and can be stopped and restarted arbitrarily on another computer. However, a task may run on a single computer at a time. A schedule is a plan describing which task will be executed on which computer at which time.

(a) Show that the duration of a schedule is at least $\max\left(\max_{1 \leqslant i \leqslant n} p_i, \frac{1}{m} \sum_{i=1}^{n} p_i\right)$.

(b) Show that there exists a schedule that achieves the bound and that it can be computed in $O(n)$ operations.

(c) Each task has now a time interval $[d_i, f_i]$ in which it must be performed. Write a linear system of constraints whose feasibility is equivalent to the existence of a schedule that runs all the tasks in the appropriate time intervals.

(d) Write an algorithm that computes a schedule obeying the time constraints and minimizing the termination time.

**Exercice 4** (Min-cut and Ising model)**.** Let $G = (V, E)$ be a nondirected weighted graph with positive weights $J_{ij}$. Each vertex $i$ has an associated scalar $h_i$. We want to solve the following optimization problem:

$$\text{maximize} \sum_{(i,j)\in E} J_{ij}\sigma_i\sigma_j + \sum_{i\in V} h_i\sigma_i$$
$$\text{such that } \sigma_i \in \{-1, 1\}.$$

Show that an optimal solution can be computed in polynomial time. (Hint: first assume that $h_i = 0$ and use a reduction to Min-cut; then refine the reduction to take $h_i$ into account.)