Advanced algorithms

Exercise sheet #4 – Parametric complexity (continued)

October 19, 2022

Exercice 1 (3-Coloring and treewidth)**.** The 3-Coloring problem is the following:

Input: A graph G;
Question: Does the vertices of G admit a coloring with 3 colors such that no two adjacent vertices have the same color?

Show that this problem is FPT with respect to the treewidth of the input. In order to do that, give an algorithm that takes as input a graph G together with a *nice* tree decomposition of width k and m nodes and that computes a 3-coloring (or ensures that there is no such coloring) in time $O(m \cdot k \cdot 3^k)$.

Exercice 2 (Quadratic kernel for MaxSat)**.** The problem MaxSat is the following:

Input: A SAT formula given in CNF with m clauses; an integer k;
Question: Is there an assignment of the variables that satisfies at least k clauses?

(a) Show that the problem always has a solution if $k \leqslant \frac{m}{2}$.

(b) Show an instance can always be reduced in polynomial time to another where no clause contains more than k literals.

(c) Show that every instance of MaxSat has a kernel of size $O(k^2)$.

Exercice 3 (Paths and colorings)**.** We consider the problem PartialHamiltonian:

Input: $G = (V, E)$ a graph; k an integer;
Output: Is there a path in G visiting exactly $\geqslant k$ vertices and no vertex more than once?

We will study in particular the case where $k = O(\log n)$, where n is the number of vertices of G.

(a) Give a naive algorithm. Is it polynomial in n when $k = O(\log n)$?

(b) Let $C : V \to \{1, \dots, k\}$ be a coloring of G with k colors (without constraints on the colors of adjacent vertices). A path is totally multicolor if every vertex has a different color.

Show how to decide the existence a totally multicolor path of length k in time $O(2^k \cdot n^2)$. *Hint: dynamic programming.*

(c) Give a probabilistic algorithm to solve PartialHamiltonian. Is the complexity polynomial when $k = O(\log n)$?

Exercice 4 (HamiltonianCycle and treewidth)**.** The HamiltonianCycle problem is the following:

Input: A graph G;
Question: Is there a cycle in G that visits all the vertices exactly once ?

Show that HamiltonianCycle is FPT with respect to the treewidth.